

Dynamic Aspects of Delta Hedging

1. Under both the discrete time Cox-Ross-Rubinstein (CRR) and continuous time Black-Scholes-Merton (BSM) models, the initial value of a so-called hedge portfolio is $V_H = \Delta S - C$; an investor buys (using borrowed money) ΔS in shares to hedge a short (written) call.¹ Under both CRR and BSM, risk is dynamically “delta” hedged from the option contract’s inception until its expiration; thus, the hedge portfolio is “delta neutral” throughout the life of the option, and its value evolves deterministically over time, in much the same way that a riskless discount bond converges over time to its terminal value.
2. Although delta hedging is not technically feasible in continuous time, it is feasible in discrete time. In the delta hedging section (Section 4) of “The Greek Letters” textbook chapter, a dynamic delta hedging numerical example is presented in Tables 2 and 3. These tables are replicated and extended in the “Dynamic Delta Hedging Spreadsheet” which is available on the Finance 4366 website. Suppose that each call option conveys the right to purchase 100 shares, and an investor seeks exposure to 100,000 shares. In order to create a “delta neutral” hedge portfolio, the investor 1) sells 1,000 call options (thus conveying the right for the option buyer to purchase 100,000 shares), 2) hedges the short option position by

¹This trading strategy creates the synthetic equivalent of lending money; reversing positions (i.e., buying rather than selling the call and shorting rather than buying the underlying) creates the synthetic equivalent of a borrowing transaction.

purchasing delta shares with borrowed money, and 3) rebalances on a weekly basis. Suppose also that the exercise price is \$50, and these options expire in 20 weeks.

3. We simulate the stock price using the GBM equation (based on an initial share price of $S = \$49$, $\Delta t = 1/52$, $\sigma = .20$, $\mu = .13$, and time to expiration of 20 weeks (thus, $T = 20/52 = .3846$)). The above referenced spreadsheet consists of two worksheets; in the worksheet labeled “Delta Hedging (ITM Close)”, the simulated stock price sequence results in the option expiring in the money, whereas in the worksheet labeled “Delta Hedging (OTM Close)” the simulated stock price sequence results in the option expiring out of the money.
4. As shown in the “Delta Hedging (ITM Close)” worksheet, when the option expires in the money, the net number of shares purchased totals 100,000, and the cost basis for these shares is \$5,195,197. Adding in interest costs of \$68,141.49, the cumulative cost of net share purchases plus interest comes to \$5,263,338.49. Since the 1,000 call options are in the money at expiration, the option buyer exercises these options and pays the investor \$5,000,000 ($\$50 \times 100,000$ shares; note that delta hedging ensures that the investor has 100,000 shares available to convey to the option buyer). Thus, the net cost of delta hedging under the “in the money” scenario comes to \$263,338. Under this scenario (as well as the “out of the money” scenario), the net cost of delta hedging 1,000 call options comes close to the initial

(\$240,000) value of the option position; any differences between the net cost of delta hedging and the initial value of the written call are due to 1) infrequent (weekly) rebalancing (as opposed to continuous rebalancing), and 2) the time value of money.

5. As shown in the “Delta Hedging (OTM Close)” worksheet, when the option expires out of the money, the net number of shares purchased is equal to zero (which makes sense because the investor does not need any shares to cover the short position). By week 20, delta goes to 0, the investor has a “naked” short position in 1,000 call options which expire worthless, and the net cost of delta hedging under the “out of the money” scenario comes to \$256,338. As noted earlier for the “in the money” scenario, the difference between the \$256,338 net cost of delta hedging under the “out of the money” and the initial \$240,000 value of the written call is due to the combined effect of infrequent rebalancing and not accounting for the time value of money.
6. The spreadsheet extends Tables 2 and 3 by enumerating weekly realized and predicted portfolio gains and losses. The realized weekly gain/loss is calculated as follows:

$$\Delta V_H = 100,000 ((C_{t-1} - C_t) + \Delta(S_t - S_{t-1})).$$

- Note that since the investor holds a short position in call options and a long position in shares, then if the share price increases (decreases) from date t to date $t-1$, the share component of the hedge portfolio ($100,000 (\Delta(S_t - S_{t-1}))$) also increase (decreases) in value, whereas the option component of the hedge portfolio ($100,000 (C_{t-1} - C_t)$) decreases (increases in value).

- The predicted weekly gain/loss is determined from an equation based on a Taylor series expansion of the weekly change in the portfolio value, and it shows the roles played by theta and gamma. As shown in the appendix,

$$\Delta V_H = \frac{\partial V_H}{\partial S} \Delta S + \frac{\partial V_H}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 V_H}{\partial S^2} \Delta S^2;$$

since delta hedging eliminates the $\frac{\partial V_H}{\partial S} \Delta S$ term, it follows that $\Delta V_H = \frac{\partial V_H}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 V_H}{\partial S^2} \Delta S^2$, where $\frac{\partial V_H}{\partial t}$ corresponds to the option's theta and $\frac{\partial^2 V_H}{\partial S^2}$ corresponds to the option's gamma.

- Not surprisingly, the realized and predicted weekly gains and losses are nearly perfectly positively correlated with each other: .992 in the in-the-money case, and .972 in the out-of-the-money case.