Determining the Stochastic Process for a Forward Contract from Ito's Lemma

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Let $F = F(S,\, t)$. Note that F is once differentiable in t and twice differentiable in S. Itô's Lemma justifies the use of the following Taylor-series-like expansion for the instantaneous change in F:

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}dS^2. \tag{1}$$

Since $dS^2 = S^2\sigma^2 dt$, substituting $S^2\sigma^2 dt$ in place of dS^2 in the Itô's Lemma equation yields equation (2):

$$\frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2}dt \tag{2}$$

Since the instantaneous change in S evolves according to the geometric Brownian motion (GBM) equation $dS = \mu S dt + \sigma S dz$, substituting $\mu S dt + \sigma S dz$ in place of dS in equation (2) yields:

$$dF = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial S}\mu S + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2}\right)dt + \frac{\partial F}{\partial S}\sigma S dz \tag{3}$$

In equation (3), we refer to $\frac{\partial F}{\partial t}$ as "theta", $\frac{\partial F}{\partial S}$ as "delta", and $\frac{\partial^2 F}{\partial S^2}$ as "gamma". Theta measures the effect that the passage of time has on F, delta measures the sensitivity of F with respect to changes in S, and gamma captures the sensitivity of delta with respect to changes in S.

Next, consider a forward contract on a non-dividend paying stock; its date t "arbitrage-free" price is $F = Se^{r(T-t)}$. Since theta $= \frac{\partial F}{\partial t} = -rSe^{r(T-t)}$, delta $= \frac{\partial F}{\partial S} = e^{r(T-t)}$, and gamma $= \frac{\partial^2 F}{\partial S^2} = 0$, we are now able to infer the stochastic process for the price of the forward contract by substituting the values for theta, delta and gamma into equation (3):

$$dF = [e^{r(T-t)}\mu S - rSe^{r(T-t)}]dt + e^{r(T-t)}\sigma S dz. \tag{4}$$

Substituting F in place of $Se^{r(T-t)}$ in equation (4), we obtain

$$dF = (\mu - r)F dt + \sigma F dz. \tag{5}$$

Note the difference in the drift rate for the forward contract vis-a-vis the drift rate for the underlying asset. Specifically, over infinitesimal units of time, the price of the forward contract grows at the rate of $(\mu - r)dt$, whereas the underlying asset grows at the rate of μdt. Intuitively this is not a surprising result, particularly in view of the fact that the "arbitrage-free" price $F = Se^{r(T-t)}$ represents the future (date T) value of the date t value of the underlying asset, compounded forward at the riskless rate of interest.